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On Normal Factor Coverings of Groups

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In this paper we extend the results on generalized nilpotent groups proved in [*1*] to other classes of groups definable by means of properties of factors of certain covering normal subgroups. We thereby obtain, in particular, new characterizations of finite supersolvable and solvable groups (Corollaries to Theorems 2 and 3).

By a *class* of groups we shall mean, as usual, a collection $\mathfrak{X}$ such that $E \in \mathfrak{X}$ and such that if $G \in \mathfrak{X}$ and $H \simeq G$, then $H \in \mathfrak{X}$. Here E is used to denote any group of order one, and e will be used for the identity element of any group. $G^{\#}$ will be used to denote the set of non-identity elements of any group G.

The crucial definitions for our purposes are as follows. If $\mathfrak{X}$ denotes some class of groups, let $\mathfrak{X}^*$ denote the class defined by $G \in \mathfrak{X}^*$ iff for each $x \in G^{\#}$ there exist subgroups $K \lhd H \lhd G$ such that $x \in H\backslash K$ and $H/K \in \mathfrak{X}$, and let $\mathfrak{X}^{\dagger}$ denote the class defined by $G \in \mathfrak{X}^{\dagger}$ iff for each $x \in G^{\#}$ there exist subgroups $K \lhd H \lhd G$ with $K \lhd G$ such that $x \in H\backslash K$ and $H/K \in \mathfrak{X}$. We shall say that K, H is a *suitable pair* for x, in either case, the precise meaning always being clear from the context. Clearly $\mathfrak{X} \leqslant \mathfrak{X}^{\dagger} \leqslant \mathfrak{X}^*$ for each class $\mathfrak{X}$.

It will be convenient to use P. Hall's closure operations. Thus $G \in Q\mathfrak{X}$, $G \in S_n\mathfrak{X}$, and $G \in P\mathfrak{X}$, will mean respectively that G is isomorphic to a quotient group of an $\mathfrak{X}$-group, subnormal subgroup of an $\mathfrak{X}$-group, and poly-$\mathfrak{X}$-group, where by a poly-$\mathfrak{X}$-group is meant a group G possessing a finite series $G = A_n \rhd A_{n-1} \rhd \cdots \rhd A_0 = E$ such that each $A_{i+1}/A_i \in \mathfrak{X}$. We shall call a series such as the last one a *normal* series; if, in addition, each $A_i \lhd G$, we call it an *invariant* series. We shall write $G \in \bar{P}\mathfrak{X}$ if G has an invariant series with $\mathfrak{X}$-factors, even though $\bar{P}$ defined in this way is not a closure operation. For each class $\mathfrak{X}$, $\bar{P}\mathfrak{X} \leqslant \mathfrak{X}^{\dagger}$; necessary and/or sufficient conditions for equality is one of our goals. Finally, $\mathfrak{M}_n$ will denote the class of groups satisfying minimal condition on normal subgroups, and $\mathfrak{M}_{sn}$ the class satisfying minimal condition on subnormal subgroups.

THEOREM 1. *If $Q\mathfrak{X} = \mathfrak{X}$, $S_n\mathfrak{X} = \mathfrak{X}$, and $G \in \mathfrak{X}^{\dagger} \cap \mathfrak{M}_n$, then each minimal normal subgroup of each epimorphic image of G is an $\mathfrak{X}$-group.*

LEMMA 1. *If $Q\mathfrak{X} = \mathfrak{X}$ and $S_n\mathfrak{X} = \mathfrak{X}$, then*

$$Q(\mathfrak{X}^\dagger \cap \mathfrak{M}_n) = \mathfrak{X}^\dagger \cap \mathfrak{M}_n$$

and

$$Q(\mathfrak{X}^* \cap \mathfrak{M}_{sn}) = \mathfrak{X}^* \cap \mathfrak{M}_{sn}.$$

Proof. Clearly the right hand side is contained in the left hand side in each case.

Assume $G \in \mathfrak{X}^\dagger \cap \mathfrak{M}_n$, $N \lhd G$, and $\bar{x} \in \bar{G}^\#$, where $\bar{x}$ denotes the image of $x \in G$ under the natural map from G onto $\bar{G} = G/N \in Q(\mathfrak{X}^\dagger \cap \mathfrak{M}_n)$. There exist $K \lhd H \lhd G$ such that $K \lhd G$, $x \in H \backslash K$, $H/K \in \mathfrak{X}$. We have $\bar{x} \in \bar{H} = HN/N$, and if $\bar{x} \notin \bar{K} = KN/N$, then $\bar{K}$, $\bar{H}$ is a suitable pair for $\bar{x}$ in $\bar{G}$, since $\bar{H}/\bar{K} \simeq HN/KN \simeq H/(KN \cap H) \simeq (H/K)/(KN \cap H)/K) \in Q\mathfrak{X} = \mathfrak{X}$, and we are finished.

Assume $\bar{x} \in \bar{K}$. We let $K_0 = K$, $H_0 = H$, and proceed inductively to define a chain of normal subgroups of G,

$$\text{(c)} \qquad H_0 > K_0 \geqslant H_1 > K_1 \geqslant H_2 > K_2 \geqslant \cdots,$$

with the property that $\bar{x} \in \bar{H}_i = H_iN/N$ and $H_i/K_i \in \mathfrak{X}$ for each i. Thus suppose $\bar{x} \in \bar{K}_i = K_iN$ for some i; such is the case for $i = 0$. Then $x = kn$ with $k \in K_i \backslash N$, $n \in N$. Choose $B \lhd A \lhd G$ such that $B \lhd G$, $k \in A \backslash B$, $A/B \in \mathfrak{X}$. Let $H_{i+1} = A \cap K_i$, $K_{i+1} = B \cap K_i$. Then $K_{i+1} \lhd H_{i+1} \lhd G$, $K_{i+1} \lhd G$, $k \in H_{i+1} \backslash K_{i+1}$, and

$$\overline{H_{i+1}}/\overline{K_{i+1}} \simeq H_{i+1}/(K_{i+1}N \cap H_{i+1}) \simeq (H_{i+1}/K_{i+1})/((K_{i+1}N \cap H_{i+1})/K_{i+1})$$

with $H_{i+1}/K_{i+1} \simeq B(A \cap K_i)/B \lhd A/B$, so that $\overline{H_{i+1}}/\overline{K_{i+1}} \in QS_n\mathfrak{X} = \mathfrak{X}$. Since $\bar{x} = \bar{k}$, we thus see $\overline{K_{i+1}}$, $\overline{H_{i+1}}$ to be a suitable pair for $\bar{x}$, unless $x \in \overline{K_{i+1}}$. Also, $H_i > K_i \geqslant H_{i+1} > K_{i+1}$. Since $G \in \mathfrak{M}_n$, the chain (c) must terminate eventually, say with $H_m = K_m$, and then necessarily $\overline{K_{m-1}}$, $\overline{H_{m-1}}$ is a suitable pair for $\bar{x}$.

Thus we have established the first equality of the lemma and the second follows similarly.

LEMMA 2. *If $S_n\mathfrak{X} = \mathfrak{X}$, $G \in \mathfrak{X}^\dagger$, and N is a minimal normal subgroup of G, then $N \in \mathfrak{X}$.*

Proof. Assume N to be minimal normal in $G \in \mathfrak{X}^\dagger$, and $x \in N^\#$. There is a suitable pair K, H for x in G, and then $K \cap N$, $H \cap N$ is a suitable pair for x in N. For $K \cap N < H \cap N \leqslant N \lhd G$ yields $K \cap N = E$ and $H \cap N = N$ by the minimality of N, and also $N/E = H \cap N/K \cap N \simeq K(H \cap N)/K \lhd H/K \in \mathfrak{X}$ so that N, $H \cap N/K \cap N \in S_n\mathfrak{X} = \mathfrak{X}$.

Proof of Theorem 1. Use Lemmas 1 and 2.

Let $\mathfrak{F}$ denote the class of all finite groups.

THEOREM 2. *If* $Q\mathfrak{X} = \mathfrak{X}$ *and* $S_n\mathfrak{X} = \mathfrak{X}$, *then* $\mathfrak{X}^\dagger \cap \mathfrak{F} = \bar{P}\mathfrak{X} \cap \mathfrak{F}$.

Proof. It suffices to show $\mathfrak{X}^\dagger \cap \mathfrak{F} \leqslant \bar{P}\mathfrak{X}$, and this we do by induction. Assume the statement false, and let G be a counterexample of minimal order. G is not simple, for otherwise the only possible suitable pair would be E, G so that $G \in \mathfrak{X} \leqslant \bar{P}\mathfrak{X}$, a contradiction. Thus G contains a nontrivial minimal normal subgroup N, which is in $\mathfrak{X}$ by Theorem 1. $G/N \in \mathfrak{X}^\dagger$ by Lemma 1, and $|G/N| < |G|$, so that $G/N \in \bar{P}\mathfrak{X}$. Therefore G, being a extension of an $\mathfrak{X}$-group by a $\bar{P}\mathfrak{X}$-group, is a $\bar{P}\mathfrak{X}$-group, contrary to our assumption.

We record a special case.

COROLLARY. *A finite group G is supersolvable* iff *for each* $x \in G^\#$ *there exist subgroups* $K \lhd H \lhd G$ *such that* $K \lhd G$, $x \in H\backslash K$, *and* H/K *is cyclic.*

LEMMA 3. *If* $S_n\mathfrak{X} = \mathfrak{X}$, *then* $S_n\mathfrak{X}^\dagger = \mathfrak{X}^\dagger$ *and* $S_n\mathfrak{X}^* = \mathfrak{X}^*$.

Proof. Assume $F \lhd\lhd G \in \mathfrak{X}^*$, $x \in F^\#$, and K, H a suitable pair for x in G. Then $K \cap F$, $H \cap F$ is a suitable pair for x in F, as $H \cap K/K \cap F \simeq K(H \cap F)/K \lhd\lhd H/K$ so that $H \cap F/K \cap F \in S_n\mathfrak{X} = \mathfrak{X}$. Thus $F \in \mathfrak{X}^*$ and so $S_n\mathfrak{X}^* \leqslant \mathfrak{X}^*$.

If $K \lhd G$ above, then also $K \cap F \lhd F$, yielding $S_n\mathfrak{X}^\dagger \leqslant \mathfrak{X}^\dagger$.

THEOREM 3. *If* $Q\mathfrak{X} = \mathfrak{X}$ *and* $S_n\mathfrak{X} = \mathfrak{X}$, *then* $\mathfrak{X}^* \cap \mathfrak{F} \leqslant P\mathfrak{X} \cap \mathfrak{F}$.

Proof. This follows by an easy induction using Lemmas 3 and 1 and the fact that any extension of a $P\mathfrak{X}$-group by a $P\mathfrak{X}$-group is again a $P\mathfrak{X}$-group.

We mention two cases of particular interest. Let π denote any set of primes. π-separable groups are discussed in [*2*], for example.

COROLLARY. *A finite group G is solvable* (*respectively* π*-separable*) iff *for each* $x \in G^\#$ *there exist subgroups* $K \lhd H \lhd G$ *such that* $x \in H\backslash K$ *and* H/K *is Abelian* (*respectively a* π*-group or a* π'*-group*).

Theorems 2 and 3 give results for the class of finite groups. In the case of $\dagger$, Theorem 1 also gives satisfactory results if finiteness is weakened to minimal condition on normal subgroups. Recall that a group G is *hyperabelian* (respectively *hypercyclic* or *Baer supersolvable*) if every epimorphic image, not E, of G possesses an Abelian (respectively cyclic) normal subgroup, not E [*5*]. Thus Theorem 1 implies to following.

COROLLARY. *A group G satisfying the minimal condition on normal subgroups* *is hyperabelian* (*respectively hypercyclic*) iff *for each* $x \in G^{\#}$ *there exist subgroups* $K \lhd H \lhd G$ *such that* $K \lhd G$, $x \in H \backslash K$ *and* H/K *is Abelian* (*respectively cyclic*).

It is natural to compare the new classes with classes defined by means of normal and invariant systems [*4*]. In particular, if $\mathfrak{a}$ denotes the class of Abelian groups, how do $\mathfrak{a}^*$ and $\mathfrak{a}^{\dagger}$ compare with *SN* and *SI*? Here an *SN* (respectively *SI*) group is one having a normal (respectively invariant) system with Abelian factors [*4*]. It is clear that each *SI* group is in the class $\mathfrak{a}^{\dagger}$, and the existence of nonstrictly simple groups [*3*] shows that there are *SN* groups not in the class $\mathfrak{a}^*$. In fact, $\mathfrak{a}^* = \mathfrak{a}^{\dagger}$. We have not been able to determine whether each $\mathfrak{a}^{\dagger}$ group is an *SN* group. This is similar to the question left unanswered in [*1*], of whether a group, each element of which is residually central (see below), is necessarily a *Z*-group.

It is well-known that the derived group of a supersolvable group is nilpotent, and that this extends to generalized supersolvable groups. The next theorem shows this result to be true for the classes introduced here, also. We recall the class of groups studied in [*1*]: $G \in \mathfrak{R}$ iff each $x \in G$ is residually central, that is, $x = e$, or else there are subgroups $K \lhd H \lhd G$ such that $x \in H \backslash K$ and $[G, H] \leq K$. Let $\mathfrak{a}_0$ denote the class of cyclic groups.

THEOREM 4. *If* $G \in \mathfrak{a}_0^{\dagger}$, *then* $G' \in \mathfrak{R}$.

Proof. If K, H is a suitable pair for $x \in G'^{\#}$, then $K \cap G'$, $H \cap G'$ is seen to be a suitable central pair for $x \in G'$ by using the proof of the finite case ([*5*], p. 231).

The main theorems above are also true for operator groups, when the obvious changes are made in the statements of the theorems. In this way one obtains, for example, by taking as an operator set all automorphisms of G, the following.

COROLLARY. *A finite group G possesses a characteristic series with cyclic factors* iff *for each* $x \in G^{\#}$ *there exist subgroups* $K < H$, *both characteristic in G, such that* $x \in H \backslash K$ *and* H/K *is cyclic.*

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